PROBABILISTIC MISCONCEPTIONS OF PRESERVICE TEACHERS

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Probabilistic misconceptions are common among students and teachers alike. This study involved 18 preservice secondary mathematics teachers and documents some of the common probabilistic misconceptions held by these preservice teachers. Data was collected through a survey involving eight probability problems and two semi-structured interviews. Findings indicate that the most common misconception held by these preservice teachers related to time axis, compound events, and availability.

Many people struggle with probabilistic misconceptions in situations of uncertainty. A misconception is more than just a mistake. Mistakes are easy to make and can be made for a number of different reasons: e.g., carelessness or distractions from another classmate. A misconception, however, is an incorrect concept held by a person that leads to a systematic pattern of errors (Khazanov, 2008). Teachers' practices and beliefs play an extremely important role in student learning (Beswick, 2006, 2008; Thompson, 1984), therefore identifying and addressing probabilistic misconceptions held by preservice teachers is important because they may pass those misconceptions on to their future students. This study examines and documents some of the probabilistic misconceptions held by preservice teachers.

Theoretical Framework

Shaughnessy (1992) argued that people need to "learn about probability and statistics just to be able to make reasonable decisions (as consumers or voters or even in choosing a career) on the basis of the mounds of data and probabilistic statements that confront them" (p. 95). Any well-informed citizen or consumer should have a good understanding of probabilistic statements – free from misconceptions. However, probabilistic misconceptions are common among students of all ages – elementary students, secondary students, college students, and even adults (Shaughnessy, 1992; Fischbein & Schnarch, 1997).

Fischbein and Schnarch (1997) explored common probabilistic misconceptions of students at various ages. A seven-item questionnaire was developed and administered to five groups of students: 20 students in fifth-grade, 20 students in seventh-grade, 20 students in ninth-grade, 20 students in eleventh-grade, and 18 preservice mathematics teachers. The seven items on the questionnaire were each related to seven common probabilistic misconceptions. Each of these is briefly discussed below (for further discussion see Fischbein & Schnarch, 1997; Kahneman, Slovic, & Tversky, 1982). Misconceptions related to the use of the representativeness heuristic involve an individual estimating the likelihood of an event based on how similar that event is to the population that it comes from. Misconceptions associated with negative and positive recency effects involve a person relating the probability of an event to the most recent similar events (e.g., flipping a coin a certain number of times). Misconceptions related to compound and simple events involve a misunderstanding of all possible outcomes for two events occurring simultaneously (e.g., rolling two dice). Misconceptions related to the conjunction fallacy involve a person believing that, under certain conditions, the probability of an event appears to be higher than the probability of the intersection of the same event with another. Misconceptions related to

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

sample size involve individuals neglecting the importance of the size of a sample when estimating probabilities. Misconceptions related to the availability heuristic involve a person estimating the likelihood of an event based on how easy it is to think of an example or recall a particular instance of the event. Finally, the time-axis fallacy involves a person inverting the time axis of cause implying effect.

Fischbein and Schnarch (1997) hypothesized that all of the probabilistic misconceptions would decrease with age, and specifically, that they would stabilize during what Piaget labeled as the formal operational period (ages 12 and above). However, the actual findings of the study were extremely varied. Some of the misconceptions decreased with age, some remained constant, some increased with age, and some were absent altogether. Misconceptions dealing with the representativeness heuristic, negative recency, and the conjunction fallacy all decreased with age. Misconceptions dealing with compound and simple events were frequent among the students in the study and stable across all ages. Misconceptions dealing with the effect of sample size, the availability heuristic, and the effect of the time axis all increased with age. Only one of the misconceptions, positive recency, seemed to be absent from the students.

In summary, probabilistic misconceptions are common among students, teachers, and citizens (Fischbein & Schnarch, 1997; Wilkins, 2007; Shaughnessy, 1992). The purpose of this study is to identify and discuss probabilistic misconceptions of secondary mathematics preservice teachers.

Methods

Participants

Eighteen preservice secondary mathematics teachers participated in the study. All of the preservice teachers were graduating seniors in a five-year teacher preparation program. The preservice teachers graduate with an undergraduate degree in mathematics after four years and a master's degree in education after one additional year. The preservice teachers' undergraduate degree in mathematics included at least one course in probability or statistics, which they had all already taken. At the time of the study all of the preservice teachers were enrolled in a senior capstone course in which one of the researchers was the instructor.

One limitation of the study is the dual relationship that one of the researchers had with the participants in the study. The researcher personally knew all of the participants because they were students in her class. However, the researcher and the participants had never before discussed probabilistic misconceptions. Participation in the study was voluntary and was in no way linked to course credit.

Data Collection

The study consisted of an initial survey and two follow-up semi-structured interviews. The initial survey was designed to identify probabilistic misconceptions held by the preservice teachers. All eighteen of the preservice teachers in the study completed the initial survey. The initial survey included eight problems, each of which focused on one of the seven common misconceptions outlined previously by Fischbein & Schnarch (1997) with two of the problems focused on the effect of sample size.

Preservice teachers completed the eight problems by selecting the correct answer from three or four answer choices. In addition, preservice teachers were asked to explain their thinking for each of the eight problems. It was extremely important to have the preservice teachers explain their problem solving process for each of the eight problems. In this way the researchers were

able to more accurately determine whether or not the preservice teachers actually held a particular probabilistic misconception.

From the results of the initial survey, six preservice teachers who demonstrated probabilistic misconceptions were selected. Of these six preservice teachers, only two of them participated in the follow-up interview sessions. The semi-structured interviews were especially beneficial because they allowed the researcher to "experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). Both of the follow-up interview sessions were video recorded for future analysis.

This first semi-structured interview consisted of a short background interview and a time for the preservice teacher to work on two probability problems. The background interview focused on the preservice teachers' prior experiences with mathematics in general and with probability and statistics in particular. The background interview also focused on the preservice teachers' attitudes about teaching probability and statistics in the future. The preservice teacher then engaged with two probability problems that related directly with two common probabilistic misconceptions, the availability heuristic and compound and simple events.

The second semi-structured interview consisted of a probability activity that dealt specifically with the representativeness heuristic. The activity combined a hands-on experiment in which the preservice teacher flipped six fair coins 50 times in order to find experimental probabilities and also created a theoretical model for the same situation. While completing the probability activity, the interviewer probed the mathematical and probabilistic understanding of the preservice teacher by asking questions such as: "What is your reasoning for this?" and "Why did you choose this approach?"

Data Analysis

The analysis of the data included both quantitative and qualitative methods. From the initial screening survey, the percentage of preservice teachers who demonstrated the various probabilistic misconceptions was calculated. The video recording from the two semi-structured interviews was transcribed and analyzed for rich evidence of probabilistic misconceptions held by the preservice teachers.

Results

The percentage of teachers who held the various probabilistic misconceptions is presented in Table 1. The most common misconception held among these preservice teachers is related to the effect of the time axis (72.2%). Interestingly, after all eighteen participants completed the initial survey they animatedly debated their answers to this particular problem. The second most common misconception held among these preservice teachers is related to compound and simple events (44.4%). The third most common misconception held among preservice teachers is related to the availability heuristic (33%). The specific problems dealing with these three probabilistic misconceptions from the initial survey can be found in Table 2 below.

Effect of time axis	72.2%
Compound and simple events	44.4%
Availability	33.3%
Effect of sample size	27.7%
Effect of sample size	22.2%
Conjunction fallacy	11.1%
Negative recency	5.5%
Positive recency	0%
Representativeness	0%

Table 1. Percent of preservice teachers who demonstrated misconceptions (N = 18)

Effect of the time axis
Jack and Jill each receive a box containing two white marbles and two black marbles. a) Jack extracts a marble from his box and finds out that it is a white one. Without replacing the first marble, he extracts a second marble. Is the likelihood that his second marble is also white smaller than, equal to, or greater than the likelihood that it is a black marble? b) Jill extracts a marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white. Is the likelihood that the first marble she extracted is white smaller than, equal to, or greater than the likelihood that it is black? Explain your thinking.
Compound and simple events
Suppose Cathy rolls two dice simultaneously. What of the following has a greater chance of happening? a) Obtaining two sixes. b) Obtaining a five and a six. c) Both have the same chance. Explain your thinking.
Availability
When choosing a committee comprised of 2 members from among 10 candidates, the number of possibilities is: a) Smaller than… b) Equal to… c) Greater than… the number of possibilities when choosing a committee of 8 members from among 10 candidates. Explain your thinking.

Table 2. Initial survey problems

Semi-Structured Interviews

For the purposes of this paper, the results from the semi-structured interviews will focus primarily on Amanda, one of the two preservice teachers who participated in the follow-up interviews. During the first interview, Amanda was given two tasks. These tasks were related to availability and compound and simple events. The misconceptions associated with these two types of problems were among the most common in the initial survey. On the initial survey, Amanda did not seem to hold a misconception related to availability, but she did seem to hold a misconception related to compound and simple events.

The first task related to availability (see Figure 1). Amanda's initial approach was to draw all the possible paths in each of the grids. This turned out to be a daunting task and led Amanda to some confusion. She eventually took a step back and tried to relate the paths in each of the grids to a more concrete example. She exclaimed,

> Trying to figure out the number of paths in the grid is like trying to pick out a pizza. If you have to choose a certain kind of crust, a certain kind of cheese, a certain kind of

> meat, and a certain kind of vegetable to put on the pizza, then you would just multiply the number of each to get the total number of possible pizzas.

Amanda related this back to the number of possible grids by saying,

> For the first *x*, you've got one, two, three, four, five, six, seven, eight possibilities to go to the next row. And then from whatever *x* you pick there, you've got eight more ways to go to the next row… yeah, that would give you all the paths because it would be just like making a tree.

This reasoning convinced Amanda that she had developed a correct method to find the possible paths in each grid by multiplying the number of rows in the grid by the number of symbols in each row. Based on Amanda's final answer to the problem and her reasoning supporting that answer, Amanda did not seem to demonstrate a misconception related to the availability heuristic.

Consider the grid below. A path is a polygonal chain of line segments, starting at the top row and proceeding to the bottom row and meeting one and only one symbol in each row.

Grid A	Grid B
xxxxxxxx	xx
xxxxxxxx	xx
xxxxxxxx	xx
	xx
	xx
	xx
	xx
	xx
	xx

a) Are there more paths possible in Grid A?
b) Are there more paths possible in Grid B?
c) Are there about the same number of possible paths in each grid?

Figure 1. Availability task

The second task related to compound and simple events (see Figure 2). For the first question in this task, Amanda's response was,

> It's one-half…it doesn't matter…this reminds me of genetics…and it doesn't matter what your first child is, I mean, we know that his first child is a son, but that has no bearing on what the [other child is]. Every time you have a child, you have a one-half probability of it being a boy or a girl. So it's one-half because they are completely independent events.

Amanda's response was especially interesting because she is a double major in both mathematics and biology. Because of her focus on biology and her past experience in genetics classes, Amanda was very passionate to say that no matter what we know about a situation (such as the gender of one child) the probability of the other child being a boy will *always* be one-half. Amanda was confident that the probability of the second child being a boy was one-half. In response to the second question in this task, she said that it still did not matter what the gender of the eldest child was. She said, "even if you ask the probability that…the other child is a girl, it's

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still one-half." Based on her response to this task and her reasoning for her response, it seems as though Amanda does hold a probabilistic misconception related to compound and simple events. More specifically, Amanda's misconception hinges on the idea that the events are "independent" of each other. Amanda's misconception also suggests that she does not take into consideration the fact that there are four different possibilities for Mr. Smith's two children; the two children could either be a boy and a boy, a boy and a girl, a girl and a boy, or a girl and a girl.

1) Mr. Smith is the father of two. We meet him walking along the street with a young boy whom he proudly introduces as his son. What is the probability that Mr. Smith's other child is also a boy?
2) We meet Mr. Smith (who we know to be the father of two) in the street with a boy. He is very elaborate with his introduction, presenting the boy as his eldest child. What is the probability that Mr. Smith's other child is also a boy?

Figure 2. Effect of time axis task

During the second interview, Amanda engaged with a task that had both an experimental and a theoretical component (see Figure 3). The hands-on experiment required that she flip six coins 50 times and record the number of heads (out of six) for each of the 50 flips. Before starting the experiment, Amanda hypothesized that "three heads would come up the most…there's just more ways to get three heads than six heads." This shows that Amanda had a notion of which combinations of heads and tails would be the most common. Also, when flipping the coins, Amanda observed, "one head is coming up more than one tail." This surprised her. She thought that she should have about the same number of flips when there was only one head showing as when there was only one tail showing. This observation shows that Amanda also understood the notion of symmetry in the distribution of the number of heads out of six.

Based on the experiment, Amanda developed an experimental probability model. This model described the probability of getting *x* number of heads when six coins were flipped at one time. These beginning observations seem to indicate that Amanda has a good understanding of the distribution and sample space she is dealing with and that she does not hold a misconceptions related to the representativeness heuristic.

<table>
<tr><td colspan="7">1) Flip 6 coins 50 times and record the number of heads in the chart below.</td></tr>
<tr><td>6 H</td><td>5 H</td><td>4 H</td><td>3 H</td><td>2 H</td><td>1 H</td><td>0 H</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="7">2) Based on your data, what is the probability of getting 6 heads? 5 heads? 4 heads? 3 heads? 2 heads? 1 head? 0 heads?</td></tr>
<tr><td colspan="7">3) Make a list of all possible outcomes for flipping 6 coins.</td></tr>
<tr><td colspan="7">3) Based on your list, develop a mathematical model to find the theoretical probability for the outcomes of flipping 6 coins.</td></tr>
</table>

Figure 3. Experimental and theoretical probability task

The theoretical component of the task required that Amanda develop a theoretical probability model, again describing the probability of getting x number of heads when six coins were flipped at once. Amanda initially used a tree diagram to list out all the possible outcomes for flipping six coins. From her tree diagram, Amanda quickly articulated that a flip of TTTTTH is very different from a flip of HTTTTT. She then explained that you could "move around" the one head to get all the possible ways that the flip of six coins could have one head. Because Amanda had an intuitive notion of symmetry, she knew that this was similar to one tail (which she also explained was the same thing as looking at five heads). Amanda also explained that there would be a lot more ways to "move around" three heads than just one head. Because it would have been difficult to count all the theoretical ways of getting x number of heads on a flip of six coins based on her tree diagram, Amanda switched to using her understanding of combinations to develop the theoretical model.

Throughout this activity, Amanda did not seem to demonstrate that she held any probabilistic misconceptions associated with the representativeness heuristic. She did demonstrate that she held intuitive notions of symmetry and combinations related to the activity based on her observations during the experiment and the ease in which she calculated her theoretical model.

Discussion and Conclusions

The purpose of this study was to explore and document some of the common probabilistic misconceptions of preservice teachers. Although all of the preservice teachers in this study are undergraduate mathematics majors who have taken at least one course in probability and statistics, these preservice teachers still demonstrate probabilistic misconceptions. The most common probabilistic misconceptions of the eighteen preservice teachers were related to the effect of the time axis, compound and simple events, and availability.

Most secondary mathematics preservice teachers are required to take at least one probability and statistics course, similar to the preservice teachers in this study. However, because data analysis and probability is one of the National Council of Teachers of Mathematics' five content standards for pre-kindergarten through grade 12 mathematics education (NCTM, 2000), it might be necessary to examine both the probability and statistics courses and methods courses of secondary mathematics preservice teachers in order to discuss any potential probabilistic misconceptions of preservice teachers.

Based on Amanda's responses during the second interview, it was clear that she valued the theoretical model over the experimental model. Amanda developed both an experimental and a

theoretical model for finding the probability of *x* number of heads when flipping six coins. Both models accurately described the probabilities. However, Amanda trusted her theoretical model over her experimental model. She compared the experimental model to the theoretical model and found that the experimental model was not "off by that much." This is most likely due to the fact that Amanda had little experience with experimental probabilities; she was much more familiar with theoretical probabilities. This is typical for many preservice teachers like Amanda; they do not often have meaningful experiences with activities in which they find both the experimental and the theoretical probabilities and compare the two models.

One benefit of a task like this – developing both an experimental and theoretical model for finding probabilities and then comparing the two models – is to show preservice teachers that *both* models are valuable. It is important to explicitly explore the connections between both models to enforce the validity, usefulness, and power of the experimental model.

Identifying and discussing probabilistic misconceptions of preservice teachers should be an important part of the teachers' methods courses as preservice teachers will have an influence on their future students' understanding of probability. Further, having preservice teachers engage in hands-on experiments involving well-designed tasks may be one way to help them address their own misconceptions (Shaughnessy, 1982; Wilkins, 2007, Khazanov, 2008) and better prepare them to help their future students develop sound probabilistic conceptions.

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